

# RECONSTRUCTION AND FINITENESS RESULTS FOR FOURIER-MUKAI PARTNERS

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## 1. ABSTRACT

We show that a scheme of finite type over a field is determined by its bounded derived category of coherent sheaves together with a collection of autoequivalences corresponding to an ample family of line bundles. In particular for a quasi-projective variety we need only a single autoequivalence. This imposes strong conditions on the Fourier-Mukai partners of a projective variety. Namely, if  $X$  is any smooth projective variety over  $\mathbb{C}$ , we have a representation  $\rho$  of  $\text{Aut}(D_{\text{coh}}^b(X))$  on  $H^*(X)$ . Now if  $\ker \rho = 2\mathbb{Z} \times \text{Pic}^0(X) \ltimes \text{Aut}^0(X)$  then we are able to conclude that  $X$  has finitely many Fourier-Mukai partners. In particular we are able to show that abelian varieties have finitely many Fourier-Mukai partners. We also show that from the derived category of coherent sheaves on a scheme of finite type over a field one can recover the full subcategory of objects with proper support. As applications we show that an abelian variety can be recovered from its derived category of coherent  $D$ -modules and that a smooth variety with the property that the canonical bundle restricted to any proper subvariety is either ample or anti-ample can be recovered from its derived category of coherent sheaves (generalizing a well-known theorem of Bondal and Orlov).

## 2. INTRODUCTION

In [4], Bondal and Orlov prove that a smooth projective variety with ample or anti-ample canonical bundle can be reconstructed from its derived category of coherent sheaves. This reconstruction uses the fact that the Serre functor corresponds to an ample or anti-ample line bundle (together with a shift). There is nothing special about the the Serre functor in this reconstruction except that it is intrinsic to the category. The starting point for this paper is to instead take the data of both the derived category and a functor which comes from an ample line bundle. Using this data, one can reconstruct the scheme. More generally, we show that a scheme of finite type over a field is determined by its derived category together with a collection of autoequivalences corresponding to an ample family of line bundles. In this way the schemes which are derived equivalent to a given scheme  $X$  are encoded in the autoequivalences of  $D_{\text{coh}}^b(X)$ . This idea can be applied to abelian varieties because the autoequivalences of their derived categories are well understood.

It is believed conjecturally that if you consider the derived category of coherent sheaves on a fixed variety there are only finitely many isomorphism classes of varieties with an equivalent derived category. We show that this conjecture holds for a variety  $X$  over a field of characteristic zero if the autoequivalences of  $D_{\text{coh}}^b(X)$  which act trivially on cohomology are precisely  $2\mathbb{Z} \times \text{Pic}^0(X) \ltimes \text{Aut}^0(X)$ . Moreover, we provide a bound in this case. It is well known that this condition holds for abelian varieties [10, 16, 9]. For abelian varieties with Neron-Severi group equal to  $\mathbb{Z}$  we calculate an explicit bound which turns out to be the number of inequivalent cusps for the action of  $\Gamma_0(N)$  on the upper half plane ( $N$  is a certain invariant of the abelian variety). Our proof of this conjecture extends the well-known fact that any abelian variety is derived equivalent to at most finitely many abelian varieties. Shortly after the posting of the original version of this paper, Huybrechts and Nieper-Wisskirchen proved that any Fourier Mukai partner of an abelian variety is in fact an abelian variety. This result is also proved in an unpublished thesis by Fabrice Rosay. The results of Huybrechts, Nieper-Wisskirchen, and Rosay, are in fact stronger than the bound we calculate in some cases.

We also show that from the derived category of coherent sheaves on a variety one can recover the full subcategory of objects with proper support. Using this category one can apply the techniques of Orlov and Bondal in [4] to show that any smooth variety with ample or anti-ample canonical bundle can be reconstructed from its derived category (eliminating the projectivity assumption). More generally we need only require that the canonical is either ample or anti-ample when restricted to any proper closed subvariety.

It has been observed by D. Arinkin that one can reconstruct an abelian variety from its derived category of  $D$ -modules using these results. It has been conjectured by Orlov that this statement is true for any variety.

### 3. RECONSTRUCTION

We begin by proving that the derived category together with an ample family of line bundles determines the scheme. Equivalently one could use the dual family of line bundles by switching to the inverse functors but we omit such statements. We also make all the statements for the bounded derived category of coherent sheaves. For affine schemes or quasi-projective schemes over a field it is known that the bounded derived category of coherent sheaves can be recovered from the derived category of quasi-coherent sheaves. It is precisely the full subcategory of locally cohomologically finitely presented objects. In fact the statement is true for a larger class of schemes satisfying a certain technical condition (see [21] for details). If one instead took the bounded derived category of quasi-coherent sheaves on a separated noetherian scheme, then once again one could recover the bounded derived category of coherent sheaves as the full subcategory of compact objects [21]. Thus we could equally well make the statements below for the derived category of quasi-coherent sheaves over an affine scheme or a quasi-projective variety or for the bounded derived category of quasi-coherent sheaves over a separated noetherian scheme but once again we omit such statements.

As a matter of convention, a variety always means an integral scheme of finite type over a field  $k$ , all half exact functors such as tensor products and pullbacks acting on objects in the derived category are taken to be derived functors unless otherwise stated, and any functor between derived categories is taken to be graded. The reconstructions below (only) necessitate the graded structure and sometimes the  $k$ -linear structure, so when we say that  $X$  can be reconstructed from  $D_{coh}^b(X)$ , we mean that  $X$  is determined by  $D_{coh}^b(X)$  as a  $(k\text{-linear})$  graded category and when we posit equivalences  $F : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$  we require that  $F$  is a  $(k\text{-linear})$  graded functor.

We begin by recalling the notion of an ample family of line bundles (see [23] 6.II 2.2.3).

**Definition 3.1.** *Let  $X$  be a quasi-compact, quasi-separated scheme and  $\{\mathcal{A}_i\}$  be a family of invertible sheaves on  $X$ .  $\{\mathcal{A}_i\}$  is called an ample family of line bundles if it satisfies the following equivalent conditions:*

- a) *The open sets  $X_f$  for all  $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$  with  $i \in I$ ,  $n > 0$  form a basis for the Zariski topology on  $X$ .*
- b) *There is a family of sections  $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$  such that the  $X_f$  form an affine basis for the Zariski topology on  $X$ .*
- c) *There is a family of sections  $f \in \Gamma(X, \mathcal{A}_i^{\otimes n})$  such that the  $X_f$  form an affine cover of  $X$ .*
- d) *For any quasi-coherent sheaf  $\mathcal{F}$  and  $i \in I$ ,  $n > 0$  let  $\mathcal{F}_{i,n}$  denote the subsheaf of  $\mathcal{F} \otimes \mathcal{A}_i^{\otimes n}$  generated by global sections. Then  $\mathcal{F}$  is the sum of the submodules  $\mathcal{F}_{i,n} \otimes \mathcal{A}_i^{\otimes -n}$ .*
- e) *For any quasi-coherent sheaf of ideals  $\mathcal{F}$  and  $i \in I$ ,  $n > 0$ ,  $\mathcal{F}$  is the sum of the submodules  $\mathcal{F}_{i,n} \otimes \mathcal{A}_i^{\otimes -n}$ .*
- f) *For any quasi-coherent sheaf  $\mathcal{F}$  of finite type there exist integers  $n_i, k_i > 0$  such that  $\mathcal{F}$  is a quotient of  $\bigoplus_{i \in I} \mathcal{A}_i^{\otimes -n_i} \otimes \mathcal{O}_X^{k_i}$ .*
- g) *For any quasi-coherent sheaf of ideals  $\mathcal{F}$  of finite type there exist integers  $n_i, k_i > 0$  such that  $\mathcal{F}$  is a quotient of  $\bigoplus_{i \in I} \mathcal{A}_i^{\otimes -n_i} \otimes \mathcal{O}_X^{k_i}$ .*

A scheme which admits an ample family of line bundles is called divisorial. All smooth varieties are divisorial (see [23] 6.II). More generally any normal noetherian locally  $\mathbb{Q}$ -factorial scheme with affine diagonal is divisorial [5].

We fix the following notation:  $X$  is a noetherian scheme,  $\{\mathcal{A}_i\}$  is a finite ample family of line bundles on  $X$ , and  $A_i$  is the autoequivalence of  $D_{coh}^b(X)$  which corresponds to tensoring with the sheaf  $\mathcal{A}_i$ . We use multi-index notation so that  $\mathcal{A}^d := \mathcal{A}_1^{\otimes d_1} \otimes \cdots \otimes \mathcal{A}_r^{\otimes d_r}$  for  $d \in \mathbb{N}^r$ .

**Definition 3.2.** *An object  $P \in D_{coh}^b(X)$  is called a point object with respect to a collection of autoequivalences  $\{\mathcal{A}_i\}$  if the following hold:*

- i)  $A_i(P) \cong P$  for all  $i$ ,
- ii)  $\text{Hom}^{<0}(P, P) = 0$ ,
- iii)  $\text{Hom}^0(P, P) = k(P)$  with  $k(P)$  a field.

**Proposition 3.3.** *Let  $\{\mathcal{A}_i\}$  be an ample family of line bundles on a noetherian scheme  $X$ . Then  $P$  is a point object with respect to  $\{\mathcal{A}_i\}$  if and only if  $P \cong \mathcal{O}_x[r]$  for some  $r \in \mathbb{Z}$  and some closed point  $x \in X$ .*

*Proof.* Any structure sheaf of a closed point is clearly a point object. On the other hand suppose  $P$  is a point object for  $\{\mathcal{A}_i\}$ . Let  $\mathcal{H}_j$  be the  $j^{\text{th}}$  cohomology sheaf of  $P$ . Consider the map  $\mu_f : P \rightarrow P \otimes \mathcal{A}^d$  given by multiplication by  $f \in \Gamma(X, \mathcal{A}^d)$ . Since  $P$  is a point object  $\mu_f$  is either 0 or an isomorphism for any  $f$ . In particular, the induced map,  $\mu_{f,j}$  on  $\mathcal{H}_j$  is either 0 or an isomorphism. If  $\mu_{f,j} = 0$  we have that  $\text{Supp}(\mathcal{H}_j) \subseteq Z(f)$  and if  $\mu_{f,j}$  is an isomorphism then  $\text{Supp}(\mathcal{H}_j) \cap Z(f) = \emptyset$ .

Now suppose that  $x, y$  are two distinct points in the support of  $\mathcal{H}_j$ . Since  $\{\mathcal{A}_i\}$  is an ample family, the opens  $X_f$  with  $f \in \Gamma(X, \mathcal{A}^d)$  form a basis for the Zariski topology of  $X$ . Thus there exists a function  $f \in \Gamma(X, \mathcal{A}^d)$  such that  $f$  vanishes on  $x$  but not on  $y$ , yielding a contradiction. It follows that  $\mathcal{H}_j$  is supported at a point. The result then follows from [10] Lemma 4.5 (this lemma uses the noetherian assumption).  $\square$

We have shown that one can recover the structure sheaves of points up to shift, now we wish to recover the line bundles up to shift. This motivates the following definition:

**Definition 3.4.** *An object  $L \in D_{\text{coh}}^b(X)$  is called invertible for a set  $S$  if for all  $P \in S$  there exists an  $n_P \in \mathbb{Z}$  such that :*

$$\text{Hom}(L, P[i]) = \begin{cases} k(P) & \text{if } i = n_P \\ 0 & \text{otherwise} \end{cases}$$

An invertible object with respect to a collection of autoequivalences  $\{\mathcal{A}_i\}$  is an invertible object for the set of point objects with respect to  $\{\mathcal{A}_i\}$ . Let  $S := \{\mathcal{O}_x[n] | x \in X \text{ is a closed point and } n \in \mathbb{Z}\}$ . It follows from the proof of Proposition 2.4 in [4] that if  $X$  is a noetherian scheme any invertible object  $L \in D_{\text{coh}}^b(X)$  for  $S$  is isomorphic to  $\mathcal{L}[t]$  for some line bundle  $\mathcal{L}$  and some  $t \in \mathbb{Z}$  (the set of shifted line bundles).

**Lemma 3.5.** *A divisorial scheme of finite type over a field can be recovered from  $D_{\text{coh}}^b(X)$  together with the full subcategory of objects with zero dimensional support. Furthermore if  $Y$  is a divisorial scheme of finite type over a field and  $F : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  is an equivalence which maps objects with zero dimensional support to objects with zero-dimensional support then  $X \cong Y$ .*

We provide two proofs of the lemma, the later proof is meant to follow more closely that found in [4] and in particular the variation of that proof found in [10].

*Proof.* Let  $S$  denote the full subcategory of objects with zero dimensional support. Consider objects  $P \in S$  satisfying:

- i)  $\text{Hom}^{<0}(P, P) = 0$ ,
- ii)  $\text{Hom}^0(P, P) = k(P)$  with  $k(P)$  a field.

Call this class of objects  $T$ . By Lemma 4.5 of [10], all such objects are isomorphic to  $\mathcal{O}_x[r]$  for some  $r \in \mathbb{Z}, x \in X$ . As noted above, invertible objects with respect to  $T$  are precisely the objects isomorphic to shifted line bundles. Let  $p_T^N := \{P \in T | \text{Hom}(N, P) = k(P)\}$  where  $N$  is a fixed invertible object. We may assume  $p_T^N = \{\mathcal{O}_x, | x \in X\}$ . We call this set  $X_0$  and proceed by recovering the Zariski topology on this set. The line bundles are now

$$l_T^N := \{L | L \text{ is invertible and } \text{Hom}(L, P) = k(P) \forall P \in p_{\{\mathcal{A}_i\}}^N\}.$$

Now given any two objects  $L_1, L_2 \in l_{\{\mathcal{A}_i\}}^N$ , and  $\alpha \in \text{Hom}(L_1, L_2)$  we get an induced map,

$$\alpha_P^* : \text{Hom}(L_2, P) \rightarrow \text{Hom}(L_1, P).$$

Then denote by  $X_\alpha$  the subset of those objects  $P \in p_{\{\mathcal{A}_i\}}^N$  for which  $\alpha_P^* \neq 0$ . Then  $X_\alpha$  is the complement of the zero-locus of  $\alpha$ . By assumption letting  $\alpha$  run over all morphisms in  $\text{Hom}(L_1, L_2)$  and  $L_1, L_2$  run over all line bundles, we get a basis for the Zariski topology on  $X_0$ . From this set it is easy to see that one can add prime ideals for each irreducible closed subset to recover  $X$  together with its Zariski topology.

From here the two proofs diverge. For each open set  $U \subseteq X$  we consider the full subcategory

$$D_U := \{A \in D_{\text{coh}}^b(X) | \text{Hom}(A, P[i]) = 0 \forall P \in U_0, i\}.$$

This is the subcategory of objects supported on  $X \setminus U$ . Localizing we reconstruct  $D_{\text{coh}}^b(U)$ . Hence we can reconstruct the structure sheaf on  $X$  as  $\mathcal{O}_X(U) := \text{Hom}_{D_{\text{coh}}^b(U)}(N, N)$ .

The second proof requires  $X$  to be a quasi-projective variety over an algebraically closed field. It proceeds as follows: for every object  $P \in p_T^N$  we can consider morphisms  $\psi \in \text{Hom}(P, P[1])$ . Then to each such morphism we get an exact triangle  $\mathcal{E}_\psi \rightarrow P \rightarrow P[1]$ .

Now consider finite dimensional vector subspaces  $V \subseteq \text{Hom}(N, M)$  for some  $M \in l_{\{\mathcal{A}_i\}}^N$  such that,

- i) For all  $P, Q \in p_T^N$  there exists  $f \in V$  such that  $f_P^* : \text{Hom}(M, P) \rightarrow \text{Hom}(N, P) \neq 0$  and  $f_Q^* : \text{Hom}(M, Q) \rightarrow \text{Hom}(N, Q) = 0$ ,
- ii) For all  $P \in p_T^N$  and all  $\psi \in \text{Hom}(P, P[1])$  there exists a  $\phi \in \text{Hom}(M, \mathcal{E}_\psi)$  such that  $\text{Hom}(N, \bullet)(\phi)(W_P) \neq 0$  where  $W_P := \{v \in V | v_P^* = 0\}$ .

The first condition says that  $V$  separates points the second says that it separates tangent vectors hence  $V$  gives an embedding into projective space and we recover the scheme structure on  $X$ .

Now when  $F : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$  is an equivalence and  $F^{-1} \circ (\bullet \otimes \mathcal{M}_i) \circ F = (\bullet \otimes \mathcal{A}_i)$ . Then  $F$  takes point objects with respect to  $\{(\bullet \otimes \mathcal{A}_i)\}$  to point objects with respect to  $\{(\bullet \otimes \mathcal{M}_i)\}$ . Since for any  $y \in Y, n \in \mathbb{Z}$  the objects  $\mathcal{O}_y[n]$  are point objects with respect to  $\{(\bullet \otimes \mathcal{M}_i)\}$  we have that  $F(\mathcal{O}_y[n]) \cong \mathcal{O}_x[r]$ . Hence we get a set theoretic map  $X \rightarrow Y$ . This map is clearly injective as  $F$  is an equivalence. Furthermore it is surjective since the collection of objects  $\{\mathcal{O}_y[n] | y \in Y, n \in \mathbb{Z}\}$  has the property that for any object  $B \in D_{coh}^b(Y)$  there exists an object  $\mathcal{O}_y[n]$  such that  $\text{Hom}(B, \mathcal{O}_y[n]) \neq 0$ . After getting a bijection, all of the above reconstructions are identical. In particular the sections of  $\{\text{Hom}(N, N \otimes \mathcal{M}_i^d) | d \in \mathbb{N}^r\}$  form a basis for the topology on  $Y$  hence  $\{(\bullet \otimes \mathcal{M}_i)\}$  is also an ample family. It follows that  $X \cong Y$ .  $\square$

**Remark** Notice that the proof applies to a larger class of schemes than just those of finite type over a field. Instead suppose  $X$  is a noetherian scheme and let  $X_0$  be the set of closed points of  $X$ . The proof requires that the prime ideals are in bijection with irreducible closed subsets of the set of closed points. Many of the proofs below apply to this situation as well.

We now arrive at an analogous theorem to that in [4],

**Theorem 3.6.** *Let  $X$  be a divisorial scheme of finite type over a field. Then  $X$  can be reconstructed from its derived category of coherent sheaves together with a collection of autoequivalences corresponding to an ample family of line bundles. Let  $Y$  be a divisorial scheme of finite type over a field,  $F : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$  an equivalence,  $\{\mathcal{A}_i\}$  an ample family of line bundles on  $X$ , and  $\{\mathcal{M}_i\}$  any collection of line bundles on  $Y$ . If  $F^{-1} \circ (\bullet \otimes \mathcal{M}_i) \circ F = (\bullet \otimes \mathcal{A}_i)$  then  $X \cong Y$ .*

*Proof.* This follows immediately from Proposition 3.3 and Lemma 3.5.

Once again we provide a second proof which resembles more closely the one found in [4]. For this proof we use the ample family of line bundles to reconstruct  $X$  as an open subset of its multigraded coordinate ring as follows. First we recover the Zariski topology on  $X$  as above, recall that this requires the choice of an invertible object  $N$ . From this object we recover the multigraded coordinate ring as  $S := \bigoplus_{d \in \mathbb{N}^r} \text{Hom}(N, A^d(N))$ . Note

that  $\forall \alpha \in S$  we have the open set  $X_\alpha$ . Now we use the open embedding of  $X$  into  $\text{Proj}(S)$  to recover the scheme structure. As we have recovered  $S$ , we can consider just the relevant  $\alpha \in S$  (as in [5]). It is a fact that  $X_\alpha$  is an open subset of  $S_\alpha$  with equality if and only if  $X_\alpha$  is affine [5]. Now for each  $x \in X$ , as  $\{x\}$  is closed, we can write  $x$  as a complement  $x = (\bigcup_{\beta \in I_x} X_\beta)^c$  for some indexing set  $I_x$ . So given a relevant  $\alpha \in S$

we have that  $X_\alpha$  is affine if and only if every closed point of  $S_\alpha$  is expressed as  $(\bigcup_{\beta \in I_x} (S_\alpha)_\beta)^c \subseteq \text{Proj}(S)$  for the indexing set corresponding to some  $x \in X_\alpha$ . This recovers the affine  $X_\alpha$  and hence we have recovered the scheme structure.  $\square$

**Remark** If  $X$  and  $Y$  are divisorial schemes of finite type over a field and  $F : D^b(X) \rightarrow D^b(Y)$  is an equivalence, then any ample family of line bundles  $\{\mathcal{A}_i\}$  on  $X$  induces a collection of autoequivalences  $\{F \circ (\bullet \otimes \mathcal{A}_i) \circ F^{-1}\} \in \text{Aut}(D_{coh}^b(Y))$ . The category  $D_{coh}^b(Y)$  together with this family of autoequivalences will reconstruct the space  $X$  via the above procedure. Thus spaces with equivalent derived categories are somehow encoded in the autoequivalences of the category. However, it is not clear to the author whether or not it is possible to give nice categorical conditions on a collection of autoequivalences that insures it comes from an ample family of line bundles.

**Corollary 3.7.** *Suppose  $X$  is a scheme of finite type over a field with ample trivial bundle i.e.  $X$  is quasi-affine. Then  $X$  can be reconstructed from  $D_{coh}^b(X)$ . If  $Y$  is any divisorial scheme of finite type over a field such that  $D_{coh}^b(X)$  is equivalent to  $D_{coh}^b(Y)$  then  $X \cong Y$ .*

*Proof.* The functor  $(\bullet \otimes \mathcal{O}_X)$  is the identity functor, hence for any equivalence  $F : D_{coh}^b(X) \rightarrow D_{coh}^b(Y)$  we have  $F^{-1} \circ (\bullet \otimes \mathcal{O}_Y) \circ F \cong (\bullet \otimes \mathcal{O}_X)$ .  $\square$

Our aim now is to reconstruct an abelian variety from its derived category of  $D$ -modules, let  $\mathfrak{P}rop(X)$  denote the full triangulated subcategory of  $D_{coh}^b(X)$  consisting of objects supported on proper subvarieties of  $X$ .

**Theorem 3.8.** *For a scheme  $X$  over a field  $k$ ,  $\mathfrak{P}rop(X)$  is equivalent to the full subcategory of  $D_{coh}^b(X)$  consisting of objects  $A \in D_{coh}^b(X)$  with the property that  $\text{Hom}(A, B)$  is finite dimensional over  $k$  for all  $B \in D_{coh}^b(X)$ . Hence as either  $k$ -linear graded or  $k$ -linear triangulated categories,  $\mathfrak{P}rop(X)$  can be recovered from  $D_{coh}^b(X)$ .*

*Proof.* Suppose  $A^\bullet$  is supported on a proper subscheme. Consider the spectral sequence,

$$E_2^{(p,q)} = \text{Hom}_{D_{coh}^b(X)}(H^{-q}(A^\bullet), B^\bullet[p]) \Rightarrow \text{Hom}_{D_{coh}^b(X)}(A^\bullet, B^\bullet[p+q]).$$

Each term in the spectral sequence is a finite dimensional vector space hence  $\text{Hom}_{D_{coh}^b(X)}(A^\bullet, B^\bullet)$  is finite dimensional.

On the other hand, suppose  $A^\bullet$  is supported on a non-proper subscheme. Let  $m$  be the greatest integer such that  $H^m(A^\bullet)$  is supported on a non-proper subscheme. Since the support is not proper, there exists an affine curve  $C \subseteq \text{Supp}(H^m(A^\bullet))$ . Let  $i$  denote the inclusion map,  $i : C \rightarrow \text{Supp}(H^m(A^\bullet))$ . We compute  $\text{Hom}_{D_{coh}^b(X)}(A^\bullet, i_* i^* H^m(A^\bullet))$  (the pullback and pushforward are not derived, we just take the sheaf concentrated in degree zero). Using the same spectral sequence as before we have,

$$E_2^{(p,q)} = \text{Hom}_{D_{coh}^b(X)}(H^{-q}(A^\bullet), i_* i^* H^m(A^\bullet)[p]) \Rightarrow \text{Hom}_{D_{coh}^b(X)}(A^\bullet, i_* i^* H^m(A^\bullet)[p+q]).$$

Notice that since  $C$  is affine  $E_2^{(0,-m)} \cong \text{Hom}(i^* H^m(A^\bullet), i^* H^m(A^\bullet))$  is the endomorphism ring of a module supported on all of  $C$ . Hence this is an infinite dimensional vector space and furthermore all the terms below it are finite dimensional. Hence  $E_\infty^{(0,-m)}$  is infinite dimensional. Thus there is a filtration of  $\text{Hom}_{D_{coh}^b(X)}(A^\bullet, i_* i^* H^m(A^\bullet)[-m])$  which contains an infinite dimensional vector space. Hence

$$\text{Hom}_{D_{coh}^b(X)}(A^\bullet, i_* i^* H^m(A^\bullet)[-m])$$

is infinite dimensional.  $\square$

**Remark** This statement is formally similar to a result of Orlov's in [18]. This result says that the category  $\mathfrak{P}erf(X)$  formed by perfect complexes can be recovered from the (unbounded) derived category of coherent sheaves. That is, the triangulated subcategory formed by perfect complexes is precisely the full subcategory of homologically finite objects.

**Corollary 3.9.** *Let  $X$  and  $Y$  be divisorial varieties. Let  $\omega_{X_{sm}}$  be the canonical bundle of the smooth locus of  $X$ . Suppose that any proper closed positive dimensional subvariety  $Z$  is contained in  $X_{sm}$  and  $\omega_{X_{sm}}$  restricted to  $Z$  is either ample or anti-ample, then  $X$  can be reconstructed from its derived category (as a  $k$ -linear graded category). Furthermore if  $D_{coh}^b(X)$  is equivalent to  $D_{coh}^b(Y)$  then  $X \cong Y$ .*

*Proof.* By Theorem 3.8 we can recover  $\mathfrak{P}rop(X)$  from  $D_{coh}^b(X)$ . Now consider all properly supported objects which are point objects with respect to the identity. Call this class  $\mathcal{C}$ . Notice that the structure sheaf of any proper subvariety is in  $\mathcal{C}$ . Suppose  $A \in \mathcal{C}$  is orthogonal to all other objects in  $\mathcal{C}$  then the support of  $A$  is not contained in any positive dimensional proper subvariety and by Lemma 4.5 of [10] it must be the structure sheaf of a point. Hence we have recovered the structure sheaves of points (up to shift) which are not contained in a positive dimensional proper subvariety. Now take the (left) orthogonal to this collection of structure sheaves in  $\mathfrak{P}rop(X)$ . This is the category of objects whose support is contained in a positive dimensional proper subvariety. This category comes equipped with a Serre functor given by  $(\cdot \otimes \omega_{X_{sm}})[\dim X_{sm}]$ . By assumption (after shifting) this functor acts as tensoring with an ample or anti-ample line bundle. It follows from the proof of Proposition 3.3 that using this functor we can recover the objects isomorphic to structure sheaves of closed points (up to shift) which are contained in a positive dimensional proper subvariety. Hence

we can recover the structure sheaves of closed points on  $X$  (up to shift). The statement follows from the proof of Lemma 3.5.  $\square$

**Theorem 3.10** (Arinkin). *An abelian variety  $A$  can be reconstructed from its derived category of coherent  $D$ -modules. If two abelian varieties  $A$  and  $B$  have equivalent derived categories of coherent  $D$ -modules then  $A \cong B$*

*Proof.* Let  $\mathfrak{g} = H^1(X, \mathcal{O}_A)$ . Then there is a tautological extension

$$0 \rightarrow \mathfrak{g}^* \otimes \mathcal{O}_A \rightarrow \mathcal{E} \rightarrow \mathcal{O}_A \rightarrow 0$$

which corresponds to the identity of  $\text{End}(\mathfrak{g}^*) = \text{Ext}^1(\mathcal{O}_A, \mathfrak{g}^* \otimes \mathcal{O}_A)$ . Let  $A^\natural$  be the  $\mathfrak{g}^*$ -principal bundle associated to the extension  $\mathcal{E}$ . Then the derived category of  $D$ -modules on the dual abelian variety  $\hat{A}$  is equivalent to the category of coherent sheaves on  $A^\natural$  [14, 22, 19].

We now show that the space  $A^\natural$  has only finite sets of points as proper subvarieties. Hence from either of the previous corollaries we can recover  $A^\natural$  and use Hodge theory to recover  $A$  then dualize to recover  $\hat{A}$ .

Suppose  $P \subseteq A^\natural$  is a proper subvariety. Since  $A^\natural$  is an affine bundle over  $A$ , the projection  $\pi : P \rightarrow A$  is finite and the pullback of  $A^\natural$  to  $P$  will have a section i.e. it will be a trivial affine bundle. Now  $A^\natural$  is represented by the ample class  $\text{Id} \in \text{End}(\mathfrak{g}^*) = \text{Ext}^1(\mathcal{O}_A, \mathfrak{g}^* \otimes \mathcal{O}_A) = H^1(A, \Omega_A^1)$ . Now since  $\pi : P \rightarrow A$  is finite, the projection of  $\pi^*(\text{Id}) \in H^1(P, \pi^*\Omega_A^1)$  onto  $H^1(P, \Omega_P^1)$  is also ample. The only way an ample class on  $P$  can be zero is if it is a finite set of points.  $\square$

**Remark** Actually over  $\mathbb{C}$ ,  $A^\natural$  is a Stein space and hence we see immediately that finite sets of points are the only proper closed subvarieties.

#### 4. AUTOEQUIVALENCES AND FOURIER-MUKAI PARTNERS

In this section we consider the case of smooth projective varieties. For projective varieties, a single ample line bundle gives an ample family. Furthermore due to a famous result of Orlov [17] generalized by Canonaco and Stellari [6], any equivalence between derived categories of smooth projective varieties is a Fourier-Mukai transform. Hence in what follows when both varieties in question are smooth and projective we say only that they are equivalent and often use the fact that the equivalence is a Fourier-Mukai transform. Now suppose that we have two varieties  $X$  and  $Y$ . Any equivalence

$$F : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$$

induces an isomorphism of the groups of autoequivalences,

$$\begin{aligned} F^* : \text{Aut}(D_{\text{coh}}^b(X)) &\rightarrow \text{Aut}(D_{\text{coh}}^b(Y)) \\ \Phi &\mapsto F \circ \Phi \circ F^{-1}. \end{aligned}$$

Suppose  $\mathcal{A}$  is an ample line bundle on  $X$ . We saw above that if  $F^*(\mathcal{A}) = (\bullet \otimes \mathcal{M})$  then  $X \cong Y$ . In fact when  $F$  is a Fourier-Mukai transform we can say more. Namely, if a Fourier-Mukai transform  $\Phi_P$  takes skyscraper sheaves of points to shifted skyscraper sheaves of points we have that  $\Phi_P \cong \gamma_* \circ (\bullet \otimes \mathcal{N})[s] := (\gamma, \mathcal{N})[s]$ , for some isomorphism  $\gamma : X \xrightarrow{\sim} Y$ ,  $\mathcal{N} \in \text{Pic}(X)$  and  $s \in \mathbb{Z}$  [10, 13]. Hence if  $F^*(\mathcal{A}) = (\bullet \otimes \mathcal{M})$  then  $F \cong (\gamma, \mathcal{N})[s]$ . For the sake of applications we also want to consider the more general situation in which  $F^*(\mathcal{A}) = (\tau, \mathcal{L})[r]$ . First we show that if  $F^*(\mathcal{A}) = (\tau, \mathcal{L})[r]$  then  $r = 0$  and  $\tau^n$  is the identity for some  $n \in \mathbb{Z}$ . Replacing  $\mathcal{A}$  by  $\underbrace{(\bullet \otimes \mathcal{A}) \circ \cdots \circ (\bullet \otimes \mathcal{A})}_{n \text{ times}}$  we get the following,

**Lemma 4.1.** *Let  $X$  and  $Y$  be a smooth projective varieties. Let  $\mathcal{A}$  be an ample line bundle on  $Y$ ,  $\tau \in \text{Aut}(X)$ , and  $\mathcal{L} \in \text{Pic } X$  and suppose we have an equivalence  $F : D_{\text{coh}}^b(X) \cong D_{\text{coh}}^b(Y)$  and  $F^*((\bullet \otimes \mathcal{A})) = (\tau, \mathcal{L})[r]$  for some  $r \in \mathbb{Z}$ . Then  $F \cong (\gamma, \mathcal{N})[s]$  for some line bundle  $\mathcal{N} \in \text{Pic}(Y)$ , an isomorphism  $\gamma : X \xrightarrow{\sim} Y$ , and  $s \in \mathbb{Z}$ .*

*Proof.* Now take an arbitrary  $A \in D_{\text{coh}}^b(X)$  then if  $r \neq 0$  by considering the homology sheaves of  $A$  we notice that  $A$  is not isomorphic to  $\tau_*(A \otimes \mathcal{L})[r] = (\tau, \mathcal{L})[r](A)$ . However for any  $y \in Y$ ,  $\mathcal{O}_y \cong (\mathcal{O}_y \otimes \mathcal{A})$ . Hence  $r = 0$ .

Let  $F \cong \Phi_P$  and  $F^{-1} \cong \Phi_Q$  for some  $P, Q \in D^b(X \times Y)$ . Then  $P \boxtimes Q \in D^b(X \times X \times Y \times Y)$  defines an equivalence  $\Phi_{P \boxtimes Q} : D^b(X \times X) \rightarrow D^b(Y \times Y)$  [10]. It follows easily from the formula for composition of

Fourier-Mukai transforms that  $\Phi_{P \boxtimes Q}(S) \cong T$  where  $F^*(\Phi_S) = \Phi_T$  with  $\Phi_S$  and  $\Phi_T$  autoequivalences. Let  $\Delta$  be the diagonal map and  $\delta_n^\tau := \tau_* \mathcal{L} \otimes \tau_*^2 \mathcal{L} \otimes \dots \otimes \tau_*^n \mathcal{L}$ . Then we have,

$$\underbrace{(\bullet \otimes \mathcal{A}) \circ \dots \circ (\bullet \otimes \mathcal{A})}_{n \text{ times}} \cong \Phi_{\Delta_* (\mathcal{A}^{\otimes n})} \text{ and } \underbrace{(\tau, \mathcal{L}) \circ \dots \circ (\tau, \mathcal{L})}_{n \text{ times}} \cong \Phi_{(\text{id} \times \tau^n)_* \delta_n^\tau}$$

Therefore  $\Phi_{P \boxtimes Q}(\Delta_* (\mathcal{A}^{\otimes n})) \cong (\text{id} \times \tau^n)_* \delta_n^\tau$  by uniqueness of the Fourier-Mukai kernel. Let  $Z_n$  denote the fixed locus of  $\tau^n$ . Then we have,

$$\begin{aligned} \text{Ext}_X^i(\mathcal{O}_X, \mathcal{A}^{\otimes n}) &\cong \text{Ext}_{X \times X}^i(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{A}^{\otimes n}) \cong \text{Ext}_{X \times X}^i(\Phi_{P \boxtimes Q}(\Delta_* \mathcal{O}_X), \Phi_{P \boxtimes Q}(\Delta_* \mathcal{A}^{\otimes n})) \cong \\ &\text{Ext}_{Y \times Y}^i(\Delta_* \mathcal{O}_Y, (\text{id} \times \tau^n)_* \delta_n^\tau) \cong H^i(Z_n, \Delta^*(\text{id} \times \tau^n)_* \delta_n^\tau). \end{aligned}$$

In particular there exists an  $n$  such that  $Z_n \neq \emptyset$ . Then for  $z \in Z_n$ ,  $\mathcal{O}_z$  is a point object with respect to  $\tau^n \circ (\bullet \otimes \delta_n^\tau)$ . Hence  $\Phi_Q(\mathcal{O}_z) \cong \mathcal{O}_x[r]$  for some  $x \in X, r \in \mathbb{Z}$ . Using [10] Corollary 6.12, we have that  $\Phi_Q \cong f_* \circ (\bullet \otimes \mathcal{N})[s]$  where  $f : U \rightarrow Y$  is a morphism defined on an open set. This morphism must be injective since  $F$  is an equivalence. Let  $v \in f(U)$ , then as  $\mathcal{O}_v \otimes \mathcal{A}^{\otimes n} \cong \mathcal{O}_v$  we have  $\tau_*^n(\mathcal{O}_{f^{-1}(v)} \otimes \delta_n^\tau) \cong \mathcal{O}_{\tau^n(f^{-1}(v))} \cong \mathcal{O}_{f^{-1}(v)}$ . Therefore  $f^{-1}(v)$  is a fixed point. Therefore the fixed locus contains  $U$ , but as it is closed and  $X$  is irreducible, the fixed locus is the whole space, i.e.  $\tau^n = \text{id}$ .  $\square$

**Remark** If  $\mathcal{A}$  is not ample then the above is not necessarily true: if  $\mathcal{P} \in D_{\text{coh}}^b(A \times \hat{A})$  is the Poincare line bundle on an abelian variety  $A$  and  $t_a$  is translation by  $a \in A$ , then  $\Phi_{\mathcal{P}}^*(t_a) = (\bullet \otimes \mathcal{L})$  where  $\mathcal{L}$  is a degree zero line bundle.

To illustrate how such a statement can be utilized to bound the number of Fourier-Mukai partners of a given variety we provide some easy corollaries here,

**Corollary 4.2.** *The number of projective Fourier-Mukai partners of a smooth projective variety is bounded by the number of conjugacy classes of maximal abelian subgroups of  $\text{Aut}(D_{\text{coh}}^b(X))$ .*

*Proof.* The Picard group is always abelian and hence the Picard group of any Fourier-Mukai partner is contained in one of these conjugacy classes (under some equivalence). If the Picard groups of two Fourier-Mukai partners,  $Y$  and  $Z$ , are contained in the same conjugacy class then by modifying an equivalence we may assume the two Picard groups lie in the same maximal abelian subgroup. In particular under a suitable equivalence we have  $\text{Pic}(Y) \subseteq \text{Aut}(D_{\text{coh}}^b(Z))$  commutes with an ample line bundle on  $Z$ . Hence any element of  $\text{Pic}(Y)$  is of the form  $(\gamma, \mathcal{N})[s]$  as an element of  $\text{Aut}(D_{\text{coh}}^b(Z))$ . In particular an ample line bundle in  $\text{Pic}(Y)$  is mapped to an element of this form.  $\square$

Using the same reasoning, one could also say,

**Corollary 4.3.** *Suppose  $X$  is a smooth projective variety such that for every  $v \in \text{Aut}(D_{\text{coh}}^b(X))$  there exists a power of  $v$  under composition which is conjugate to  $(\gamma, \mathcal{N})[s]$  for some  $\gamma \in \text{Aut}(X)$ ,  $\mathcal{N} \in \text{Pic}(X)$ , and  $s \in \mathbb{Z}$ . Then  $X$  has no non-trivial Fourier-Mukai partners.*

*Proof.* Suppose  $Y$  is a smooth projective variety and  $F : D_{\text{coh}}^b(Y) \cong D_{\text{coh}}^b(X)$ . Let  $\mathcal{A}$  be an ample line bundle on  $Y$ . Then by hypothesis there exists an  $n$  such that  $F \circ ((\bullet \otimes \mathcal{A}^n)) \circ F^{-1} = t^{-1} \circ (\gamma, \mathcal{N})[s] \circ t$ , for some  $t \in \text{Aut}(D_{\text{coh}}^b(X))$ . So we have  $(F \circ t)^*((\bullet \otimes \mathcal{A}^n)) = (\gamma, \mathcal{N})[s]$ .  $\square$

For example, for all projective varieties with ample or anti-ample canonical we have that  $\text{Aut}(D_{\text{coh}}^b(X)) = \mathbb{Z} \times \text{Aut}(X) \ltimes \text{Pic}(X)$  where  $\mathbb{Z}$  acts by the shift functor, the proof of this statement can be found in [4] or can be seen directly the fact that the Serre functor commutes with all autoequivalences. Likewise we see that we can reconstruct such varieties as our earlier reconstruction theorem is just a generalization of the result in [4]. In any case, we see that such varieties have no non-trivial Fourier-Mukai partners. The ideas of the above corollaries lead us to our main result but first we need a lemma,

**Lemma 4.4.** *Let  $X$  be a smooth projective variety over  $\mathbb{C}$  and  $\rho$  be the representation of  $\text{Aut}(D_{\text{coh}}^b(X))$  on  $H^*(X, \mathbb{Q})$ . The image of  $\rho : \text{Aut}(D_{\text{coh}}^b(X)) \rightarrow \text{Gl}(H^*(X, \mathbb{Q}))$  is an arithmetic group.*

*Proof.* The Fourier-Mukai autoequivalences act on the topological K-theory of the space [13]. Topological K-theory is a finitely generated abelian group and its image under the Mukai vector map is a full sublattice of  $H^*(X, \mathbb{Q})$ . Hence the image of  $\rho$  preserves this full sublattice.  $\square$

We are now ready to prove our main result,

**Theorem 4.5.** *Let  $X$  be a smooth projective variety over  $\mathbb{C}$  and  $\rho$  be the representation of  $\text{Aut}(D_{\text{coh}}^b(X))$  on  $H^*(X, \mathbb{Q})$ . If  $\ker \rho = 2\mathbb{Z} \times \text{Pic}^0(X) \ltimes \text{Aut}^0(X)$  then the number of projective Fourier-Mukai partners of  $X$  is bounded by the number of conjugacy classes of maximal unipotent subgroups of  $\rho(\text{Aut}(D_{\text{coh}}^b(X)))$ . In particular since  $\text{im } \rho$  is an arithmetic group it is finite.*

*Proof.* First observe that given any equivalence of categories between  $F : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  the conjugation  $F^*$  induces an isomorphism of exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker \rho_X & \longrightarrow & \text{Aut}(D_{\text{coh}}^b(X)) & \xrightarrow{\rho_X} & \text{im } \rho_X \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \ker \rho_Y & \longrightarrow & \text{Aut}(D_{\text{coh}}^b(Y)) & \xrightarrow{\rho_Y} & \text{im } \rho_Y \longrightarrow 0 \end{array}$$

In particular we have an isomorphism  $F^* : \ker \rho_X \xrightarrow{\sim} \ker \rho_Y$ . A theorem of Rouquier (see [10, 20]) states that  $F^*$  induces an isomorphism of algebraic groups  $F^* : \text{Pic}^0(X) \ltimes \text{Aut}^0(X) \xrightarrow{\sim} \text{Pic}^0(Y) \ltimes \text{Aut}^0(Y)$ . The theorem is also proved by Rosay in his thesis. Hence the condition that  $\ker \rho = 2\mathbb{Z} \times \text{Pic}^0(X) \ltimes \text{Aut}^0(X)$  is true for all Fourier-Mukai partners of  $X$ .

Let  $Y$  and  $Z$  be two projective Fourier-Mukai partners of  $X$  with ample line bundles  $\mathcal{A}_Y$  and  $\mathcal{A}_Z$  and fix equivalences  $F : Y \rightarrow X, G : Z \rightarrow X$ . Notice that the action of an ample line bundle on the cohomology of a space is given by multiplication by the Chern character and thus is unipotent. Hence  $\rho(F^*(\bullet \otimes \mathcal{A}_Y)) := y$  and  $\rho(G^*(\bullet \otimes \mathcal{A}_Z)) := z$  are both unipotent. Now suppose they lie in the same conjugacy class of maximal unipotent subgroups. Then by altering one of the equivalences by an autoequivalence, we may assume they lie in the same maximal unipotent subgroup. Now the lower central series of any unipotent group terminates, in particular the commutator  $[y^{-1}, [y^{-1}, [y^{-1}, \dots [y^{-1}, z]]] = 1$ . Let  $b := [y^{-1}, [y^{-1}, \dots [y^{-1}, z]]$  so that we have  $y^{-1}byb^{-1} = 1$  or  $byb^{-1} = y$  pulling back to  $\text{Aut}(D_{\text{coh}}^b(Y))$  and fixing  $\rho^{-1}(b) := B$  this reads  $B^*(\bullet \otimes \mathcal{A}_Y) = (\mathcal{L}, \tau)[2r]$  for some  $\mathcal{L} \in \text{Pic}(Y)$ ,  $\tau \in \text{Aut}(Y)$ , and  $r \in \mathbb{Z}$ . Hence by Lemma 4.1  $B \cong (\mathcal{N}, \gamma)[s]$  for some line bundle  $\mathcal{N} \in \text{Pic}(Y)$ , an isomorphism  $\gamma : X \xrightarrow{\sim} Y$ , and  $s \in \mathbb{Z}$ . But we could write the same thing for the next step. Iterating this process we get that  $(\bullet \otimes \mathcal{A}_Z)$  maps to an element of the form  $(\mathcal{N}', \gamma')[s']$  under the equivalence. Applying Lemma 4.1 one more time we get  $Y \cong Z$  hence the result. The fact that the number of conjugacy classes of maximal unipotent subgroups of an arithmetic group is finite is well-known<sup>1</sup>.  $\square$

**Remark** A weakness of this result is that many important examples do not satisfy the hypotheses. For example on an even dimensional variety the square of a spherical twist acts trivially on cohomology [24] and similarly any  $\mathbb{P}^n$ -twist acts trivially on cohomology [12]. However one may be able to overcome this problem. For example, if one could show that for any projective variety  $X$  there exists a splitting,  $s$ , of  $\text{Aut}(D_{\text{coh}}^b(X))/\text{Pic}^0(X) \ltimes \text{Aut}^0(X) \rightarrow \text{im } \rho$  such that there exists an ample line bundle  $\mathcal{A}$  with  $(\bullet \otimes \mathcal{A}) \in s(\text{im } \rho)$  then the result would hold for all projective varieties. Or perhaps if one could show such a result for certain types of kernels e.g. those generated by squares of spherical twists and  $2\mathbb{Z} \times \text{Pic}^0(X) \ltimes \text{Aut}^0(X)$ , then the result would hold for those varieties with those types of kernels.

We now apply our theorem to the case of abelian varieties. The autoequivalences of the derived category of an abelian variety have been satisfactorily described in [9] and [16]. It is this understanding of autoequivalences that allows us to declare the number of Fourier-Mukai partners of an abelian variety to be finite and give an explicit bound in the case where the Neron-Severi group of the abelian variety is  $\mathbb{Z}$ .

**Theorem 4.6.** *Let  $A$  be an abelian variety over  $\mathbb{C}$ . Then the number of Fourier-Mukai partners of  $A$  is finite. Furthermore suppose that the Neron-Severi group of  $A$  is  $\mathbb{Z}$ . Let  $\mathcal{L}$  be a generator of the Neron-Severi group of  $A$  and  $\mathcal{M}$  be a generator of the Neron-Severi group of  $\hat{A}$ . As ample bundles,  $\mathcal{L}$  and  $\mathcal{M}$  induce isogenies  $\Phi_{\mathcal{L}}$  and  $\Phi_{\mathcal{M}}$  moreover  $\Phi_{\mathcal{L}} \circ \Phi_{\mathcal{M}} := N \cdot \text{Id}$ . Then the number of smooth projective Fourier-Mukai partners of  $A$  is bounded by  $\sum_{d|N} \phi(\gcd(d, \frac{N}{d}))$ . If  $N$  is square free then all projective Fourier-Mukai partners are abelian varieties and the bound is attained, i.e. the number of such partners is just  $\sum_{d|N} \phi(\gcd(d, \frac{N}{d})) = \sum_{d|N} 1 = 2^s$*

where  $s$  is the number of prime factors of  $N$ .

<sup>1</sup>Stated this way the result can be found as [15] Corollary 9.38. It is equivalent to Theorem 9.37 of [15] which says that for any parabolic subgroup  $P$ , and arithmetic group  $\Gamma$  of an algebraic group  $G$  the double-coset space  $\Gamma \backslash G_{\mathbb{Q}} / P_{\mathbb{Q}}$  is finite. The latter statement can be found for example in [2] Theorem 15.6 or [25] Theorem 13.26.



*Proof.* The conditions of Theorem 4.5 are satisfied for abelian varieties [10, 16, 9]. We show how to calculate the bound when  $\text{NS}(A) = \mathbb{Z}$ . Let  $U(A) := \{M \in \text{Aut}(A \times \hat{A}) \mid M^{-1} = \det(M)\overline{M}^{-1}\}$  denote the Polishchuk group. For simplicity we start with the case  $\text{End}(A) = \mathbb{Z}$ , so that  $U(A) = \Gamma_0(N)$ . For  $N = 1$  we let  $\Gamma_0(1) := \text{Sl}_2 \mathbb{Z}$ , so that this case is also included. For abelian varieties,  $\text{im } \rho$  is commonly notated as  $\text{Spin}(A)$  we use this convention.

Now we reduce the study of maximal unipotent subgroups of  $\text{Spin}(A)$  to maximal unipotent subgroups of  $\Gamma_0(N)$  as follows. For an abelian variety we have the following diagram [9, 10],

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathbb{Z}/2\mathbb{Z} & & \\
 & & & & \downarrow & & \\
 0 & \longrightarrow & 2\mathbb{Z} \times A \times \hat{A} & \longrightarrow & \text{Aut}(D^b(A)) & \xrightarrow{\rho} & \text{Spin}(A) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \mathbb{Z} \times A \times \hat{A} & \longrightarrow & \text{Aut}(D^b(A)) & \longrightarrow & \Gamma_0(N) \longrightarrow 0 \\
 & & \downarrow & & & & \\
 & & \mathbb{Z}/2\mathbb{Z} & & & & \\
 & & \downarrow & & & & \\
 & & 0 & & & & 
 \end{array}$$

We have an isomorphism of algebras  $\text{End}(H^*(A)) \cong \text{Cl}(\Lambda, Q)$  where  $\Lambda := H_1(A) \oplus H_1(\hat{A})$  and  $Q$  is the canonical quadratic form [9]. Hence  $F^*((\bullet \otimes \mathcal{A}))$  for some ample line bundle  $\mathcal{A}$  corresponds to  $1 + N \in \text{Cl}(\Lambda, Q)$ . This induces an action on  $\Lambda$  given by  $v \mapsto (1 + N)v(1 - N + N^2 - N^3 + \dots)$ . Hence we also have a unipotent element of  $\text{Aut}(A \times \hat{A}) = \Gamma_0(N)$ . This element is non-trivial since the kernel of the map  $\text{Spin}(A) \rightarrow \Gamma_0(N)$  is just  $\{\pm 1\}$ . Taking powers of this element yields a collection of non-trivial conjugacy classes of unipotent matrices in  $\Gamma_0(N)$ . Now suppose that two unipotent elements of  $\text{Spin}(A)$  are conjugate in  $\Gamma_0(N)$  then they are conjugate up to sign in  $\text{Spin}(A)$  but this means they are conjugate because the negative of a unipotent is not unipotent. Furthermore the maximal unipotent subgroups of  $\Gamma_0(N)$  are infinite cyclic. Therefore if two unipotent elements of  $\text{Spin}(A)$  lie in the same conjugacy class of maximal unipotent subgroup in  $\Gamma_0(N)$  some power of them is conjugate and hence this is also true in  $\text{Spin}(A)$ . Thus the number of maximal unipotent subgroups of  $\text{Spin}(A)$  is bounded by the number of maximal unipotent subgroups of  $\Gamma_0(N)$ .

Now a matrix in  $\Gamma_0(N)$  is unipotent if and only if the trace is 2 if and only if the action of the matrix on the upper half plane is parabolic. Such a matrix fixes a unique cusp on the boundary of the half plane. We identify two such cusps  $z_1 \sim z_2 \Leftrightarrow \exists \gamma \in \Gamma_0(N)$  such that  $\gamma(z_1) = z_2$ . Now suppose  $z_1 \sim z_2$  then for a unipotent matrix,  $B \in \Gamma_0(N)$ ,  $B$  fixes  $z_1$  if and only if  $\gamma^{-1}B\gamma$  fixes  $z_2$ . Notice also that all powers of  $B$  fix  $z_1$ . Hence the number of conjugacy classes of unipotent matrices together with all their powers is in bijection with the number of classes of cusps. This is a well studied phenomenon. The number of such classes of cusps is precisely the bound given,  $\sum_{d|N} \phi(\gcd(d, \frac{N}{d}))$  ([7], pg. 103).

If we allow complex multiplication we note that the condition to be in  $U(A)$  is that  $A^{-1} = \det(A)\overline{A}^{-1}$ . For a unipotent matrix, this just means the matrix is real. Since  $\text{NS}(A) = \mathbb{Z}$  being real means that we have integer entries, therefore unipotent matrices can be taken to lie in  $\Gamma_0(N)$ .  $\square$

## 5. AN ALTERNATIVE APPROACH

In a similar vein, we can weaken the condition of being ample. Here we consider what happens when we conjugate  $\sigma$ -ample line bundles. Let  $X$  be a scheme,  $\sigma \in \text{Aut}(X)$ .

**Definition 5.1.** An invertible sheaf  $\mathcal{L}$  is called right  $\sigma$ -ample if for any coherent sheaf  $\mathcal{F}$ ,

$$H^q(X, \mathcal{F} \otimes \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{m-1}}) = 0$$

for  $q > 0$  and  $m \gg 0$ .

**Definition 5.2.** An invertible sheaf  $\mathcal{L}$  is called left  $\sigma$ -ample if for any coherent sheaf  $\mathcal{F}$ ,

$$H^q(X, \mathcal{L} \otimes \mathcal{L}^\sigma \otimes \dots \otimes \mathcal{L}^{\sigma^{m-1}} \otimes \mathcal{F}^{\sigma^m}) = 0$$

for  $q > 0$  and  $m \gg 0$ .

For a divisor  $D$  on  $X$  we say that the divisor is right (resp. left)  $\sigma$ -ample if  $\mathcal{O}_X(D)$  is.

**Lemma 5.3** (Artin-Van den Bergh Lemma 4.1 or Keeler Lemma 2.5.12). *Let  $D$  be a divisor on  $X$ . Given a positive integer  $m$ ,  $D$  is right  $\sigma^{-1}$ -ample if and only if  $D + \sigma D + \dots + \sigma^{m-1} D$  is right  $\sigma^{-m}$ -ample.*

We have the following results due to Keeler,

**Theorem 5.4** (Keeler Corollary 3.5.1). *For any projective scheme right and left  $\sigma$ -ampleness are equivalent.*

So on a projective scheme we simply refer to this property as  $\sigma$ -ample.

**Theorem 5.5** (Keeler Theorem 3.5.2). *Let  $X$  be a projective scheme with an automorphism  $\sigma$ . A divisor  $D$  is  $\sigma$ -ample if and only if the action of  $\sigma$  on  $A_{\text{num}}^1(X)$  is quasi-unipotent i.e. all its eigen values are roots of unity and  $D + \sigma D + \dots + \sigma^{m-1} D$  is ample for some  $m$ .*

Now we prove a lemma which is also essentially due to the work of Keeler,

**Lemma 5.6.** *Let  $\mathcal{F}$  be a coherent sheaf on a projective variety  $X$  with  $\dim \text{Supp}(\mathcal{F}) = n = \dim X$ . Let  $\mathcal{L}$  be a  $\sigma$ -ample line bundle, for some automorphism  $\sigma$  of  $X$  and  $k+1$  be the rank of the largest Jordan block of the action of  $\sigma$ . Then  $\chi(\mathcal{F} \otimes \delta_n^\sigma)$  is a numerical polynomial,  $p(x)$ , with  $k+n \leq \deg p(x) \leq k(n-1) + n$  for  $\dim n \geq 2$ .*

*Proof.* We know this is true for the trivial bundle by a theorem of Keeler ([8] Lemmas 3.6.16 and 3.6.17). Let  $\text{ch}(\mathcal{F}) := \alpha_0 + \dots + \alpha_n$  with  $\alpha_i \in A^{n-i}(X)$ . Now by Hirzebruch-Riemann-Roch we have,

$$\chi(X, \mathcal{F} \otimes \delta_n^\sigma) = \int [\text{ch}(\mathcal{F}) \text{ch}(\delta_n^\sigma) \text{todd}(X)] = \sum_i \int [\alpha_i \text{ch}(\delta_n^\sigma) \text{todd}(X)]$$

As  $\dim \text{Supp}(\mathcal{F}) = n$ , we have  $\alpha_0 \neq 0$  hence the term  $\int [\alpha_0 \text{ch}(\delta_n^\sigma) \text{todd}(X)]$  is just a multiple of the numerical polynomial for the trivial bundle. The rest of the terms are sums of restrictions to subvarieties of lower degrees and again due to Keeler ([8] Lemma 3.6.12) these are numerical polynomials of lower degrees.  $\square$

**Proposition 5.7.** *Let  $X$  be a variety,  $Y$  be a smooth projective variety, and  $F : D_{\text{coh}}^b(X) \rightarrow D_{\text{coh}}^b(Y)$  be a Fourier-Mukai equivalence. Further let  $\mathcal{L}$  be any line bundle and  $\mathcal{A}$  be a  $\sigma$ -ample line bundle, with  $\sigma$  any automorphism of  $Y$ . Suppose further that for any subvariety  $Z \subseteq X$  such that  $\sigma : Z \rightarrow Z$ , the restriction remains  $\sigma$ -ample. If  $F^*(\bullet \otimes \mathcal{L}) = (\sigma, \mathcal{A})$  then  $F \cong (\bullet \otimes \mathcal{N}) \circ \gamma_*[s]$  for some line bundle  $\mathcal{N} \in \text{Pic}(Y)$ , an isomorphism  $\gamma : X \xrightarrow{\sim} Y$ , and  $s \in \mathbb{Z}$ .*

*Proof.* For any point  $x \in X$ ,  $\mathcal{O}_x \cong \mathcal{O}_x \otimes \mathcal{L}^m$  therefore  $F(\mathcal{O}_x) \cong \sigma_*^m(F(\mathcal{O}_x)) \otimes \delta_n^\sigma$  for any  $m$ . It follows that the homology sheaves  $\mathcal{H}^i := \mathcal{H}^i(F(\mathcal{O}_x))$  have the same property. Now consider a sheaf  $\mathcal{F}$  with the above property, we now want to show that  $\mathcal{F}$  is supported in dimension zero.

First notice that  $\sigma_*^{-m} \mathcal{F} \cong \mathcal{F} \otimes \sigma^{-m} \delta_m^\sigma$ . But  $\sigma_*^{-m} \delta_m^\sigma$  is just  $\delta_m^{\sigma^{-1}}$ . Furthermore  $\mathcal{A}$  is also  $\sigma^{-1}$  ample as they are equivalent on a projective scheme [8]. We observe that,

$$\chi(\mathcal{F}) = \chi(\sigma_*^{-m} \mathcal{F}) = \chi(\mathcal{F} \otimes \delta_m^{\sigma^{-1}})$$

Now since  $\mathcal{F} \cong \sigma_*^{-m} \mathcal{F} \otimes \delta_m^\sigma$  the support of  $\mathcal{F}$  is invariant under  $\sigma$ , hence  $\sigma$  restricts to an automorphism of  $\text{Supp } \mathcal{F}$ . We apply the previous lemma and get that this numerical polynomial must be a constant. Thus  $\mathcal{H}^i$  is supported in dimension zero. Hence by [10] Lemma 4.5,  $F(\mathcal{O}_x)$  is a shifted skyscraper sheaf. The result follows from Proposition 3.1.  $\square$

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